Application of First Order-Richardson Method to Systems of Linear Equations with Fuzzy Data

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***Abstract*—In this paper, we will applied the Richardson iterative method for solving the fuzzy linear equations. In addition, we explain the efficiency of suggested method by solving a numerical example as a fuzzy version of examples from classical circuit analysis.**

***Index Terms*—Fuzzy linear system; Fuzzy number; Richardson iterative method; Circuit Analysis.**

I. Introduction

Systems of linear equations find many real world applications in different areas scientifics, In this background, several iterative method are developed because of their simplest implementation and their lest cost of computational complexity. The fuzzy set term first appeared in 1965 when Professor Lotfi A. Zadeh of Berkeley University, USA, published an article entitled Fuzzy Sets. He has since achieved many major theoretical advances in the field and was quickly accompanied by many researchers developing theoretical work.

In this paper, we think over a fuzzy linear system with an arbitrary fuzzy number in parametric form and with a crisp coefficient matrix. Consequently, there is hugely of the numerical iterative methods for the resolution of the fuzzy linear systems such as: Gauss-Seidel (G-S), Jacobi (J), and Successive Over Relaxation (SOR) iterative method[4,5]. Next, a first order Richardson iterative method is presented for the fuzzy linear systems nonsingular.

This manuscript will begin with a fundamental construct of fuzzy number operation is bringed In Section 2. In Section 3, the principal Section of the paper, a Richardson approach is for solving crisp and fuzzy linear system. The suggested idea is shown by solving a numerical example in the Section 4. Finally,we will conclude our paper with a small conclusion given in Section 5.

II. Preliminaries

A. Fuzzy sets on the real line - Fuzzy numbers

We begin this section with some preliminary results for fuzzy sets :

• The fuzzy sets on the real line $\mathbb{R}$ are caracteresed by theirs membership functions $\tilde{u}:\mathbb{R}\to[0,1]$ ([2], [9], [8] and [14]).

• we define by $[\tilde{u}]^\alpha := \{x\in\mathbb{R}/\tilde{u}(x)\geq\alpha\}$ and $[\tilde{u}]^0 := cl(\{x\in\mathbb{R}/\tilde{u}(x)>0\})$ The α-cut ($\alpha\in[0,1]$) on the fuzzy set $\tilde{u}$ on $\mathbb{R}$

whither the closure of the set X is denotes by cl(X).

• A fuzzy set $\tilde{u}$ is called convex if

$$\tilde{u}(\lambda x+(1-\lambda)y)\geq min(\tilde{u}(x),\tilde{u}(y)),\quad x,y\in\mathbb{R},\quad \lambda\in[0,1]$$

Let $\mathbf{F}(\mathbb{R})$ denotes de family of all fuzzy sets on $\mathbb{R}$.
We tell that $\tilde{u}\in\mathbf{F}(\mathbb{R})$ is a fuzzy number, if and only if : its membership function is defined as : (1) $\tilde{u}$ is normal.
(2) $\tilde{u}$ is convex.
(3) $\tilde{u}$ is upper semicontinuous.
(4) The α-cut $[\tilde{u}]^0$ is compact.

The set of all possible fuzzy numbers $\tilde{u}$ shall be called the fuzzy-number power set $\mathcal{F}(\mathbb{R})$ with the property $\mathcal{F}(\mathbb{R})\subset\mathbf{F}(\mathbb{R})$.

• Particular fuzzy numbers :
i)A popular fuzzy number is triangular fuzzy number $\tilde{a}$ defined by a triplet $[a,\alpha,\beta]$. As well as

$$\tilde{a}(x)=\begin{cases}1+\frac{x-a}{\alpha} & ;\quad a-\alpha\leq x\leq a,\\ 1+\frac{a-x}{\beta} & ;\quad a\leq x\leq a+\beta.\end{cases}$$

ii) A trapezoidal fuzzy number $\tilde{a}$ can be expressed as $[a_L,a_U,\alpha,\beta]$ and its membership function is defined as :

$$\tilde{a}(x)=\begin{cases}1+\frac{x-a_L}{\alpha} & ;\quad a_L-\alpha\leq x\leq a_L,\\ 1 & ;\quad a_L\leq x\leq a_U.\\ 1+\frac{a_U-x}{\beta} & ;\quad a_U\leq x\leq a_U+\beta.\end{cases}$$

• The definition of addition, soustraction and scalar multiplication on $\mathcal{F}(\mathbb{R})$ are as follows :
For $\tilde{u}$,$\tilde{v}\in\mathcal{F}(\mathbb{R})$ and $\lambda\geq0$,

$$(\tilde{u}+\tilde{v})(x):=\sup_{x_1,x_2\in\mathbb{R}/x_1+x_2=x} min(\tilde{u}(x_1),\tilde{v}(x_2))\qquad(1)$$

$$(\tilde{u}-\tilde{v})(x):=\sup_{x_1,x_2\in\mathbb{R}/x_1-x_2=x} min(\tilde{u}(x_1),\tilde{v}(x_2))\qquad(2)$$

$$(\lambda\tilde{u})(x):=\begin{cases}\tilde{u}(x/\lambda) & \text{if}\quad \lambda\in\mathbb{R}-\{0\},\\ 1_{\{0\}}(x) & \text{if}\quad \lambda=0.\end{cases}\qquad(3)$$

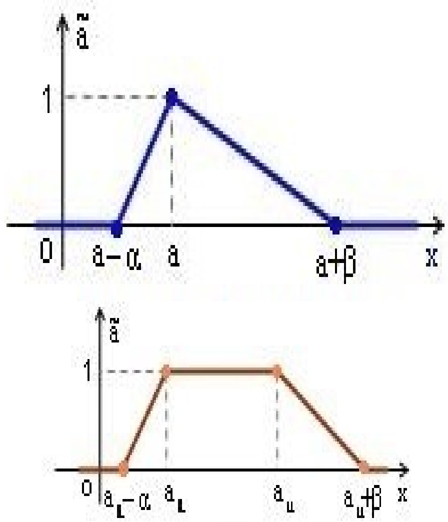

Fig. 1: Triangular and Trapezoidal Fuzzy Number

where $1_{\{0\}}$ is an indicator.
we denote by K the set of all nonempty compact subset of R and by K_C the subsets of K consisting of nonempty convex compact sets. Recall that

$$\rho(A,B) = \min_{a\in A} \|x-a\|$$

is the distance of the point $x \in R$ from $A \in K$, and that the Hausdorff separation $\rho(A,B)$ of $A, B \in k$ is defined as $\rho(A,B) = \max_{a\in A}\rho(a,B)$.
The Hausdorff metric d_H on K is defined by

$$d_H(A,B) = max\{\rho(A,B), \rho(B,A)\}$$

.
The prolongement of this metric, the Hausdorff metric on $\mathcal{F}(\mathbb{R})$ is defined by:

$$d_\infty(\tilde{u},\tilde{v}) = \sup\{d_H(\tilde{u}[\alpha],\tilde{v}[\alpha]) : 0 \leq r \leq 1\}$$

B. Fuzzy linear systems

Definition 2.1:
The $n\times n$ linear system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= y_1,\\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= y_2,\\ a_{31}x_1 + a_{32}x_2 + ... + a_{3n}x_n &= y_1, \end{aligned} \quad (4)$$

$$a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = y_n,$$

These sets of equations will be defined by the following matrix:

$$AX = Y \quad (5)$$

.
where the coefficient matrix $A = a_{ij}$, $1 \leqslant i,j \leqslant n$ is a crisp $n\times n$ matrix and $y_i \in E^1$, $1 \leqslant i \leqslant n$ This system is called a fuzzy linear system (FLS).

C. Solution of fuzzy linear systems

For arbitrary $x = (\underline{x}(r), \bar{x}(r))$, $y = (\underline{y}(r), \bar{y}(r))$ and real number $k > 0$, the usual arithmetic operations of fuzzy numbers, can be represented like this

1) $\underline{(x+y)}(r) = \underline{x}(r) + \underline{y}(r)$
2) $(x \bar{+} y)(r) = \bar{x}(r) + \bar{y}(r)$.
3) $(k\bar{x})(r) = k\bar{x}(r)$.
4) $\underline{(kx)}(r) = k\underline{x}(r)$.

Definition 2.2:
We say that a fuzzy number vector $(x_1, x_2, ..., x_n)^t$ given by $x_i = (\underline{x^i}(r), \bar{x_i}(r))$, $1 \leqslant i \leqslant n$, $0 \leqslant r \leqslant 1$, is called a solution of the FSLE if

$$\underline{\sum_{j=1}^n a_{ij}x_j} = \sum_{j=1}^n \underline{a_{ij}x_j} = \underline{y_i},$$

$$\sum_{j=1}^{n} \bar{a}_{ij}x_j = \sum_{j=1}^{n} \bar{a_{ij}}x_j = \bar{y_i}, \quad (6)$$

Either the ith equation of the system (5):
$a_{i1}(\underline{x_1}, \bar{x_1}) + ... + a_{ii}(\underline{x_i}, \bar{x_i}) + ... + a_{in}(\underline{x_n}, \bar{x_n}) = (\underline{y_i}, \bar{y_i})$,
we have

$$\begin{aligned} \underline{a_{i1}x_1} + ... + \underline{a_{ii}x_i} + ... + \underline{a_{in}x_n} &= \underline{y_i}(r) \quad (7)\\ \bar{a}_{i1}x_1 + ... + \bar{a}_{ii}x_i + ... + \bar{a}_{in}x_n &= \bar{y}_i(r),\\ 1 \leqslant i \leqslant n, 0 \leqslant r &\leqslant 1. \end{aligned}$$

as what you see from (8) for any i we have two linear systems that there can be prolonged to a $2n\times 2n$ crisp linear system as follows :

$$SX = Y \quad (8)$$

$$\rightarrow \begin{bmatrix} T\geqslant 0 & H\geqslant 0\\ H\geqslant 0 & T\geqslant 0 \end{bmatrix} \begin{bmatrix} \underline{X}\\ -\bar{X} \end{bmatrix} = \begin{bmatrix} \underline{Y}\\ -\bar{Y} \end{bmatrix}.$$

where the nonnegative entries of A are contained in T, and the absolute values of the non posetif entries are contained in H and $A = T - H$ furthermore assume that $T = L_1 + D_1 + U_1$. Thus we have $S = L + D + U$, where:

L=$\begin{bmatrix} L_1 & 0\\ H & L_1 \end{bmatrix}$, D=$\begin{bmatrix} D_1 & 0\\ 0 & D_1 \end{bmatrix}$, U=$\begin{bmatrix} U_1 & H\\ 0 & U_1 \end{bmatrix}$

We now notice that the system of linear equation (9) is a $2n\times 2n$ crisp linear system,then we can say that this system can be solved in a unique way for X if and only if the matrix S is invertible.
Yet that even though if the original matrix A is not singular, S may be.
The next results point out the difficulties for getting the fuzzy solution for a linear system.

Theorem 2.3: If the matrices $A = T - H$ and $T + H$ are invertible, then The matrix S is also invertible.

Definition 2.4:
we note by X=$\{(\underline{x_i}(r), \bar{x_i}(r)), 1 \leqslant i \leqslant n\}$ the unique solution of SX = Y.

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Let the fuzzy number vector U= $\{(\underline{u_i}(r), \bar{u}_i(r)), 1 \leqslant i \leqslant n\}$ denote by

$$\underline{u_i}(r) = min\{\underline{x_i}(r), \bar{x}_i(r), \underline{x_i}(1)\} \qquad (9)$$
$$, \bar{u}_i(r) = max\{\underline{x_i}(r), \bar{x}_i(r), \underline{x_i}(1)\},$$

Theorem 2.5:
Let S be invertible, if S^{-1} is nonnegative, then the unique solution X of equation (9) is a fuzzy vector for arbitrary vector Y continually.

III. The first-order Richardson iterative method

A. The stationary Richardson method

Consider the following linear system

$$Ax = b \qquad (10)$$

The stationary first-order Richardson's method is a simplest iterative method together with a local parameter α for the speed of iteration process. Its scheme is written like this:

$$x^{(k+1)} = x^{(k)} + \alpha r^{(k)}, \quad k \geq 0 \qquad (11)$$

Here $r^{(k)}$ is the residual vector of the current iterate :

$$r^{(k)} := b - x^{(k)}$$

The following results are inspired from [8] and [10].

Theorem 3.1:
The stationary Richardson scheme is convergent iff $\frac{2Re(\lambda_i)}{\alpha|\lambda_i|^2} \succ 1$, for all i=1,...,n, and $\lambda_i \in \mathbb{C}$ denotes the eigenvalues of the matrix A.

Theorem 3.2: Assume that, A has non negative real eigenvalues, orderly in a following way $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \succ 0$. Then, the stationary Richardson method (12) is convergent iff $0 \prec \alpha \prec \frac{2}{\lambda_1}$. furthermore, the optimal value of α is $\alpha_{opt} = \frac{2}{\lambda_1+\lambda_2}$.

Corollary 3.3: In order that matrix A is symmetric positive definite. Then the stationary Richardson method is convergent.

Remark 1: Despite of its convergence, the stationary Richardson's method has the inconvenience of being numerically unsteady.

B. The nonstationary Richardson method

More generally, permitting α in (12) according to on the iteration index, the nonstationary Richardson method or semi-iterative method is defined by

$$x^{(k+1)} = x^{(k)} + \alpha_k r^{(k)}, \quad k \geq 0 \qquad (12)$$

The euclidean space $\mathbb{R}^n$ is equipped by the canonical scalar product <.,.> and $||.||$ denotes its corresponding norm.
Since $r^{(k+1)} = r^{(k)} - \alpha A r^{(k)}$, it follows

$$||r^{(k+1)}||^2 = ||r^{(k)}||^2 - 2\alpha < r^{(k)}, Ar^{(k)} > + \alpha^2 ||Ar^{(k)}||^2$$

The optimal acceleration parameter α can be dynamically computed at every step k by

$$\frac{d}{d\alpha}||r^{(k+1)}||^2 = 0$$

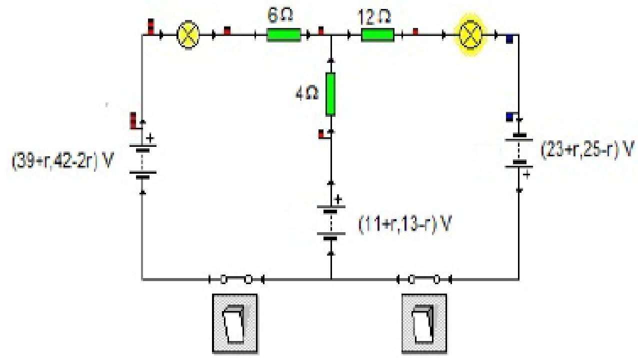

Fig. 2: A electrical circuit with fuzzy current and fuzzy source

We obtain the minimizer parameter by the next formula

$$\alpha_{opt} = \frac{< r^{(k)}, Ar^{(k)} >}{||Ar^{(k)}||^2} \qquad (13)$$

Remark 2: The nonstationary Richardson method using (14) to rating the acceleration parameter, is equally named the gradient method with dynamic parameter.

Theorem 3.4: Let A be a symmetric and positive definite matrix; then the nonstationary Richardson method is convergent for each choice of the initial data $x^{(0)}$.

IV. Numerical example in Circuit Analysis

Consider a straightforward resistive circuit with fuzzy current and a source fuzzified.

This circuit has a system of equations given by:

$$10\tilde{I}_1 - 4\tilde{I}_2 = (39 + r, 42 - 2r) - (11 + r, 13 - r)$$
$$-4\tilde{I}_1 + 16\tilde{I}_2 = (11 + r, 13 - r) + (23 + r, 25 - r). \qquad (14)$$

We can simplify system as:

$$\begin{cases} 10\tilde{I}_1 - 4\tilde{I}_2 = (26 + 2r, 31 - 3r) \\ -4\tilde{I}_1 + 16\tilde{I}_2 = (34 + 2r, 38 - 2r) \end{cases}$$

Therefore $S_1 = \begin{bmatrix} 10 & 0 \\ 0 & 16 \end{bmatrix}$ $S_2 = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}$

and then $S = \begin{bmatrix} 10 & 0 & 0 & -4 \\ 0 & 16 & -4 & 0 \\ 0 & -4 & 10 & 0 \\ -4 & 0 & 0 & 16 \end{bmatrix}$, $\tilde{Y} = \begin{bmatrix} 26 + 2r \\ 34 + 2r \\ 31 - 3r \\ 38 - 2r \end{bmatrix}$

$$S^{-1} = \begin{bmatrix} \frac{1}{9} & 0 & 0 & \frac{1}{36} \\ 0 & \frac{5}{72} & \frac{1}{36} & 0 \\ 0 & \frac{1}{36} & \frac{1}{9} & 0 \\ \frac{1}{36} & 0 & 0 & \frac{5}{72} \end{bmatrix} \quad \tilde{I} = \begin{bmatrix} \frac{71}{18} + \frac{1}{6}r \\ \frac{29}{9} + \frac{1}{18}r \\ \frac{79}{18} - \frac{5}{18}r \\ \frac{121}{36} - \frac{1}{12}r \end{bmatrix}$$

$$\tilde{I}_1 = (\tfrac{71}{18} + \tfrac{1}{6}r, \tfrac{79}{18} - \tfrac{5}{18}r),\ \tilde{I}_2 = (\tfrac{29}{9} + \tfrac{1}{18}r, \tfrac{121}{36} - \tfrac{1}{12}r)$$

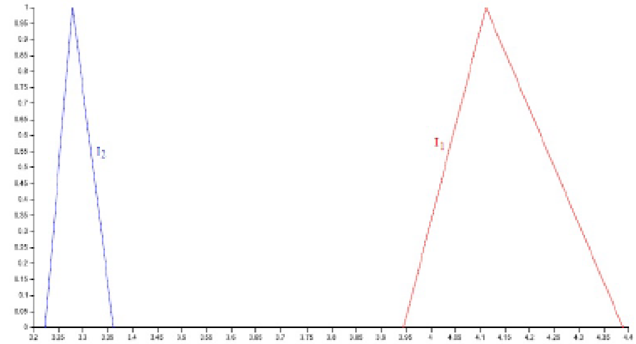

Fig. 3: The exact solution of the system in example

In our case, the matrix A is symmetric definite positive with eigenvalues 8 and 18. Next, the optimal of the minimizer parameter is $\frac{2}{18+8}$

Using the Richardson successive recursion, we obtain the approximate solution given by

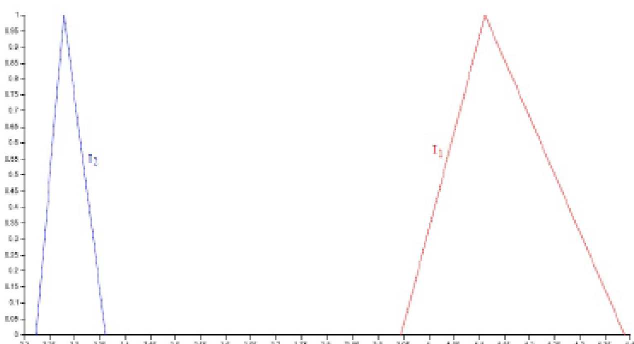

Fig. 4: The approximate solution of the system in example

V. Conclusion

In this this work, we presented the first-order Richardson approach applied to crisp systems of linear equations. Next, we adapted this method to fuzzy systems of linear equations. Finally, a pratical example of circuit analysis is given to highlight the utility of iterative methods in the fuzzy framework.

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